Date: Tuesday Dece

Examiner: S. Shahabi.

Name: ____________________

- Print your name and st
- All questions are to be space for your answer,
- No book, notes, graphi
- You are only permitted
- Show all your work and
- This examination book

THIS EXAMINATION I

$$\left(\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \quad -1 < t < 1 \right)$$

(1) **[5 pts.]** Expand $f(x) = \dfrac{1}{x^3 - 3x^2 + 3x}$ in powers of $(x-1)$, and determine the interval of convergence of the power series. **Hint:** the denominator is equal to $1 - (1-x)^3$.

$$f(x) = \frac{1}{1-(1-x)^3} = 1 + (1-x)^3 + (1-x)^6 + (1-x)^9 + \cdots$$
$$= 1 - (x-1)^3 + (x-1)^6 - (x-1)^9 + \cdots$$

It is convergent $\iff -1 < (1-x)^3 < 1$

$\iff -1 < 1-x < 1$

$\iff -1 < x-1 < 1$

Hence: $\begin{cases} R = 1, & \text{radius} \\ I =]0, 2[, & \text{interval} \end{cases}$

(2) **[5 pts.]** Find the *exact value* of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \sqrt{3}(2n+1)}$. **Recall:** $\arctan(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$.

$$L = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \cdot \sqrt{3}(2n+1)} \overset{m=n+1}{=\!=} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{3^{m-1} \cdot 3^{\frac{1}{2}}(2m-1)}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \cdot \left(\frac{1}{\sqrt{3}}\right)^{2m-1} \overset{(*)}{=} \arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}}$$

[Also note:

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2} - \arctan(\sqrt{3})$$

using: $\left(\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \quad x > 0 \right.$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \pi/6$$

(3) **[4 pts.]** If $-1 \le a < 0$, show $\left|\sum_{n=1}^{\infty} \frac{a^n}{n^2+1} - \sum_{n=1}^{N} \frac{a^n}{n^2+1}\right| < \frac{1}{(N+1)^2}$. **Hint:** a^n i

Recall: for alternating series $S = \sum_{1}^{\infty}$

have $|S - S_N| < b_{N+1}$.

Applying this inequality, we get

$$\left| \sum_{1}^{\infty} \frac{(-1)^n (-a)^n}{n^2+1} - \sum_{1}^{N} \frac{(-1)^n(-a)^n}{n^2+1} \right| < \frac{(-a)^N}{(N+1}$$

$(0 \leq -a \leq 1)$

(4) **[6 pts.]** Find the arc-length parametrization of the curve $\vec{\mathbf{r}}(t) = \left\langle \frac{2t}{t^2+1}, \pi, \frac{t^2}{t^2}\right.$

$s(t) = \int_0^t \|\vec{r}\,'(u)\| \, du$ $\qquad$ $\vec{r}\,'(u) = \langle \frac{2}{(}$

$= \int_0^t \frac{2}{1+u^2} \, du$ $\qquad$ $\|\vec{r}\,'(u)\| = \frac{}{(}$

$= 2 \left[\arctan(u)\right]_0^t$

$= 2\arctan(t) \quad (t \geq 0)$

$\longrightarrow \arctan(t) = \frac{1}{2}s \longrightarrow t = \tan(\frac{1}{2}s)$

$\longrightarrow \vec{r} = \vec{r}(s) = \left\langle \frac{2\tan(\frac{1}{2}s)}{1+\tan^2(\frac{1}{2}s)}, \pi, \frac{\tan^2(\frac{1}{2}s}{1+\tan^2(}\right.$

$= \langle \sin(s), \pi, -\cos(s) \rangle$. $\quad$ (s: th

(RmK: This is a circle (of radius 1) in a plane pa

Sp

(Rm

Exp

(x,y)

$\sqrt{|si}$

$-\frac{1}{2}$

Send

theo

$\frac{\partial}{\partial}$

or (

or

$\frac{\partial z}{\partial x_i}$

$\Rightarrow \frac{\partial^2}{\partial}$

$\Rightarrow$

- Ove

f(x

(∴ f

- Crit

lins

Si

(11) **[5 pts.]** Find $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v}$$

$= (2x + y\cos(x$

(12) **[6 pts.]** Compute

$I = \int_1^2 \left(\int_1^3 \frac{}{1+} \right.$

$= \int_1^2 (\ln(y+$

$= \left[(y+4)(\ln(\right.$

$= \{ 6(\ln(6)$

$= 6\ln(6) - 4$

$= \ln\left(\frac{6^6 \cdot 3^3}{5^5 \cdot 4^4}\right.$

I

(13) **[6 pts.]** Compute the integral $\iint_D xy\sqrt{x^2+y^2}\,dA$ over the set $D = \{(x,y)|$

$$I = \int_0^{\pi/2}\left(\int_0^1 r\cos(\theta)\, r\sin(\theta)\sqrt{r^2}\cdot r\, dr\right)$$

$$= \int_0^{\pi/2} \sin(\theta)\cos(\theta)\left[\frac{1}{5}r^5\right]_0^1 d\theta$$

$$= \frac{1}{10}\int_0^{\pi/2} \sin(2\theta)\, d\theta$$

$$= \frac{1}{10}\left(-\frac{1}{2}\cos(2\theta)\right)_0^{\pi/2} = -\frac{1}{20}\left(\cos(\pi\right.$$

$$= \boxed{\frac{1}{10}}$$

(14) **[8 pts.]** Consider the solid S (in $\mathbb{R}^3$) bounded by the planes $x + 2y + 3z = 6$

(i) Setup a *double-iterated* integral which gives the volume of S; **Don't eval**

$z = f(x,y) = \frac{1}{3}(6 - x - 2y)$

$R:\ 0 \leq x \leq 2,\quad x \leq y \leq \frac{1}{2}(6-x)$

$$V = \int_0^2\left(\int_x^{\frac{1}{2}(6-x)} \frac{1}{3}(6-x-2y)\, dy\right)dx = \iint_R f\, d$$

(ii) Setup a *triple-iterated* integral which gives the same volume; **Don't eval**

$$V = \iiint_D 1\, dV = \int_0^2\left(\int_x^{\frac{1}{2}(6-x)}\left(\int_0^{\frac{1}{3}(6-x-2y)} 1\right.\right.$$

(15) **[5 p**
$x^2 +$

$$A = \int_{\pi/6}^{5\pi/6} (\ $$

$$\begin{cases} x^2+y^2=1 \\ x^2+(y-1)^2=1 \end{cases}$$

(16) **[12**
Cylin
cone
$\rho\cos$

(i) *V*

(ii) V

(iii)