

DAWSON COLLEGE
Mathematics Department
Final Examination
Linear Algebra

201-NYC-05 Sections 01, 02, 03, 04, 05, 06

December 19th, 2017

9:30-12:30

Student Name Solutions

Student I.D. # _____

Teacher _____

Instructors:

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Instructions:

- **Print your name and student ID number in the space provided above.**
- **All questions are to be answered directly on the examination paper in the provided space.**
- **Translation and regular dictionaries are permitted.**
- **The Sharp EL-531XG or EL-531X calculators are permitted.**
- **This examination consists of 17 questions.**
- **This exam booklet must be returned intact.**

Question #	Mark
1	/5
2	/5
3	/5
4	/4
5	/6
6	/4
7	/8
8	/8
9	/4
10	/8
11	/4
12	/5
13	/12
14	/8
15	/4
16	/6
17	/4
Total	/100

1. [4+1 marks]

a) Determine the general solution of the following system using Gauss-Jordan Elimination:

$$\begin{cases} x_1 + 2x_3 + x_4 = -1 \\ 2x_1 + x_2 + 3x_3 + 3x_4 = 2 \\ -x_1 + x_2 - 3x_3 = 5 \\ 3x_1 + 2x_2 + 7x_3 + 2x_4 = -4 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 2 & 1 & 3 & 3 & 2 \\ -1 & 1 & -3 & 0 & 5 \\ 3 & 2 & 7 & 2 & -4 \end{array} \right] \begin{array}{l} r_2: r_2 - 2r_1 \\ \rightarrow \\ r_3: r_3 + r_1 \\ r_4: r_4 - 3r_1 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 2 & 1 & -1 & -1 \end{array} \right] \begin{array}{l} r_3: r_3 - r_2 \\ \rightarrow \\ r_4: r_4 - 2r_2 \\ (r_3 \leftrightarrow r_4) \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 3 & -2 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} r_3: \frac{1}{3}r_3 \\ \rightarrow \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} r_1: r_1 - 2r_3 \\ r_2: r_2 + r_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\boxed{\begin{array}{l} x_1 = 5 - 3t \\ x_2 = 1 \\ x_3 = -3 + t \\ x_4 = t \end{array} \quad t \in \mathbb{R}}$$

b) Find the particular solution when $x_1 = 2$

$$\begin{aligned} 2 &= 5 - 3t \\ -3 &= -3t \\ \Rightarrow \boxed{t = 1} \end{aligned}$$

$$\boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 1 \\ x_3 = -2 \\ x_4 = 1 \end{array}}$$

2. [5 marks] Determine the conditions on a such that the system has

a) no solution b) one solution c) infinitely many solutions.

$$x + 4y - 10z = 1$$

$$-3x + 2y + 2z = 2$$

$$4x + 2y + (a^2 - 13)z = a$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 1 \\ -3 & 2 & 2 & 2 \\ 4 & 2 & a^2-13 & a \end{array} \right] \begin{array}{l} r_2: r_2 + 3r_1 \\ r_3: r_3 - 4r_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -10 & 1 \\ 0 & 14 & -28 & 5 \\ 0 & -14 & a^2+27 & a-4 \end{array} \right] \begin{array}{l} r_3: r_3 + r_2 \end{array} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 1 \\ 0 & 14 & -28 & 5 \\ 0 & 0 & a^2-1 & a+1 \end{array} \right]$$

a) no sol'n: $a = 1$

b) one sol'n: $a \neq \pm 1, a \in \mathbb{R}$

c) infinitely many sol'ns: $a = -1$

3. [3+2 marks] Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Evaluate (wherever possible):

a) $A^{-1}BC = -1 \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

2×2 2×4 4×4

$= -1 \begin{bmatrix} 5 & -14 & -3 & 17 \\ -3 & 8 & 1 & -10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$= -1 \begin{bmatrix} 5 & -28 & -6 & 51 \\ -3 & 16 & 2 & -30 \end{bmatrix}$

$= \begin{bmatrix} -5 & 28 & 6 & -51 \\ 3 & -16 & -2 & 30 \end{bmatrix}$

b) $C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (\frac{1}{2}) & 0 & 0 \\ 0 & 0 & (\frac{1}{2}) & 0 \\ 0 & 0 & 0 & (\frac{1}{3}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$

4. [4 marks] Solve for X when:

$$\begin{bmatrix} -13 & 1 \\ 27 & 2 \end{bmatrix} X = \begin{bmatrix} -11 & \frac{1}{2} \\ 29 & 1 \end{bmatrix} X - \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\left(\begin{bmatrix} -13 & 1 \\ 27 & 2 \end{bmatrix} - \begin{bmatrix} -11 & \frac{1}{2} \\ 29 & 1 \end{bmatrix} \right) X = - \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & \frac{1}{2} \\ -2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & \frac{1}{2} \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$$

$$= - \begin{bmatrix} 1 & -\frac{1}{2} \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} = - \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$$

5. [3+3 marks]

a) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & -5 & 0 \\ 2 & 4 & 1 \end{bmatrix}$; express A^{-1} as a product of elementary matrices

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & -5 & 0 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{r_2: r_2 + 3r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{r_3: r_3 - 2r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1: r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = I \therefore A^{-1} = E_3 E_2 E_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Prove that if A is an invertible matrix and B has the same reduced row echelon form as A , then B is also invertible

A is invertible $\Rightarrow I$ is the reduced row echelon form of A

if B has the same reduced echelon form as A
 $\Rightarrow I$ is the reduced row echelon form of B
 $\Rightarrow B$ is invertible

6. [4 marks] Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & k & 1 \\ 2 & 4 & k \end{bmatrix}$

Determine the values of k for which the homogeneous system $Ax = 0$ has only the trivial solution

trivial soln whenever $\begin{vmatrix} 1 & 2 & 0 \\ 1 & k & 1 \\ 2 & 4 & k \end{vmatrix} = 1(k^2 - 4) - 2(k - 2) \neq 0$

$$k^2 - 4 - 2k + 4 \neq 0$$

$$k(k - 2) \neq 0$$

trivial soln: $k \neq 0, k \neq 2, k \in \mathbb{R}$
whenever

7. [4+4 marks] Let A, B, C be 4×4 matrices such that $\det(A) = 2$; $\det(B) = 3$; $\det(C) = -1$

Evaluate the following:

a) $\det(2A^T(3B)^{-1}C^{22}) = \det(2A^T) \det[(3B)^{-1}] \det(C^{22})$

$$= 2^4 \det(A) \cdot \frac{1}{\det(3B)} \cdot [\det(C)]^{22}$$

$$= 2^4 \cdot 2 \cdot \frac{1}{3^4 \det(B)} \cdot (-1)^{22}$$

$$= \frac{2^5}{3^4 \cdot 3} = \frac{2^5}{3^5} = \boxed{\frac{32}{243}}$$

b) $\det(AB^{-1} - A \operatorname{adj}(B))$

$$= \det(A(B^{-1} - \operatorname{adj}(B))) = \det(A) \det(B^{-1} - \operatorname{adj}(B))$$

$$= 2 \det(B^{-1} - \det(B) \cdot B^{-1})$$

$$= 2 \det(B^{-1} - 3B^{-1})$$

$$= 2 \det(-2B^{-1})$$

$$= 2(-2)^4 \det(B^{-1})$$

$$= \frac{2^5}{\det(B)} = \boxed{\frac{32}{3}}$$

8. [5+3 marks] For the following system:

$$\begin{cases} x + 2y = 1 \\ x + 3y + z = -2 \\ 2x + 4y + 3z = 0 \end{cases}$$

a) Find A^{-1} using $\text{adj}(A)$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 5 & -1 & -2 \\ -6 & 3 & 0 \\ 2 & -1 & 1 \end{bmatrix}^T$$

$$\det(A) = 3$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 5 & -6 & 2 \\ -1 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

b) Use your answer from part a) to determine the solution to the system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -6 & 2 \\ -1 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -5/3 \\ 2/3 \end{bmatrix}$$

9. [4 marks] For the following system, use Cramer's rule to solve for z only (no marks will be given if Cramer's rule is not used):

$$x - y + 4z = 10$$

$$-2x + y + z = 0$$

$$4x - y + 2z = 6$$

$$\det(A) = \begin{vmatrix} 1 & -1 & 4 \\ -2 & 1 & 1 \\ 4 & -1 & 2 \end{vmatrix} = 4(-2) - 1(3) + 2(-1) = -13$$

$$\det(A_z) = \begin{vmatrix} 1 & -1 & 10 \\ -2 & 1 & 0 \\ 4 & -1 & 6 \end{vmatrix} = 10(-2) - 0(3) + 6(-1) = -26$$

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-26}{-13} = \boxed{2}$$

10. [4+4 marks] Let $A(2,0,1)$, $B(4,-1,2)$, $C(1,2,1)$ be three points in \mathbb{R}^3 , determine the following:

a) the area of the triangle determined by the two vectors $(\vec{AB} \times \vec{AC})$ and \vec{BC}

$$\vec{AB} = (2, -1, 1), \quad \vec{AC} = (-1, 2, 0), \quad \vec{BC} = (-3, 3, -1)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ -1 & 2 & 0 \end{vmatrix} = (-2, -1, 3)$$

$$(\vec{AB} \times \vec{AC}) \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 3 \\ -3 & 3 & -1 \end{vmatrix} = (-8, -11, -9)$$

$$\|(\vec{AB} \times \vec{AC}) \times \vec{BC}\| = \sqrt{64 + 121 + 81} = \sqrt{266}$$

$$\text{Area}_{\Delta} = \boxed{\frac{\sqrt{266}}{2} \text{ units}}$$

a) the angle between $(\vec{AB} \times \vec{AC})$ and \vec{BC}

$$\text{angle} = \cos^{-1} \left(\frac{(\vec{AB} \times \vec{AC}) \cdot (\vec{BC})}{\|(\vec{AB} \times \vec{AC})\| \|\vec{BC}\|} \right) = \cos^{-1} \left(\frac{(-2, -1, 3) \cdot (-3, 3, -1)}{\sqrt{14} \cdot \sqrt{19}} \right)$$

$$= \cos^{-1} \left(\frac{0}{\sqrt{14}\sqrt{19}} \right) = \cos^{-1}(0) = \boxed{\pi \text{ radians or } 90^\circ}$$

or

$\vec{AB} \times \vec{AC}$ is a normal to the plane containing points A, B & C \therefore orthogonal to \vec{BC}

11. [4 marks]

Let $\vec{s} = (\vec{v} - \vec{w})$ and $\vec{t} = (\vec{u} \times \vec{v}) + (\vec{v} \times \vec{w}) + (\vec{w} \times \vec{u})$. Show that \vec{s} and \vec{t} are orthogonal

$$\begin{aligned}\vec{s} \cdot \vec{t} &= \\ (\vec{v} - \vec{w}) \cdot [(\vec{u} \times \vec{v}) + (\vec{v} \times \vec{w}) + (\vec{w} \times \vec{u})] \\ &= \vec{v} \cdot (\vec{u} \times \vec{v}) + \vec{v} \cdot (\vec{v} \times \vec{w}) + \vec{v} \cdot (\vec{w} \times \vec{u}) \\ &\quad - [\vec{w} \cdot (\vec{u} \times \vec{v}) + \vec{w} \cdot (\vec{v} \times \vec{w}) + \vec{w} \cdot (\vec{w} \times \vec{u})] \\ &= 0 + 0 + \vec{v} \cdot (\vec{w} \times \vec{u}) - [\vec{w} \cdot (\vec{u} \times \vec{v}) + 0 + 0] \\ &= \vec{v} \cdot (\vec{w} \times \vec{u}) - \vec{w} \cdot (\vec{u} \times \vec{v}) \\ &= \vec{v} \cdot (\vec{w} \times \vec{u}) + \vec{w} \cdot (\vec{v} \times \vec{u}) \\ &= \vec{v} \cdot (\vec{w} \times \vec{u}) - \vec{v} \cdot (\vec{w} \times \vec{u}) \\ &= 0\end{aligned}$$

\therefore vectors are orthogonal

12. [3+2 marks] Given the plane $-2x + 3y = -z - 17$, the point $A(4, -2, 2)$, and the line

$$\begin{cases} x = -1 - 2t \\ y = 4 - t \\ z = -5 + 6t \end{cases} \quad t \in \mathbb{R}$$

a) Find the point of intersection between the line and the plane

$$-2(-1 - 2t) + 3(4 - t) + (-5 + 6t) = -17$$

$$2 + 4t + 12 - 3t - 5 + 6t = -17$$

$$7t = -26 \quad t = -\frac{26}{7}$$

$$x = -1 + \frac{52}{7} = \frac{45}{7}$$

$$y = 4 + \frac{26}{7} = \frac{54}{7}$$

$$z = -5 - \frac{156}{7} = -\frac{191}{7}$$

$$\left(\frac{45}{7}, \frac{54}{7}, -\frac{191}{7} \right)$$

b) Find the general equation of the plane containing the point A and parallel to the given plane

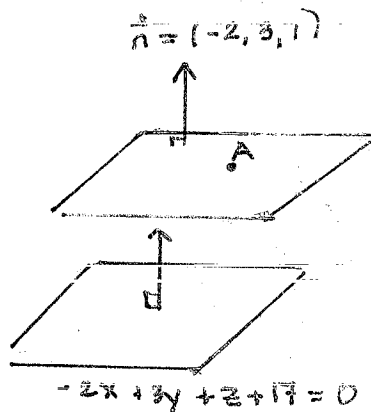
$$\vec{n} = (-2, 3, 1) \quad A(4, -2, 2)$$

eq'n:

$$-2(x-4) + 3(y+2) + (z-2) = 0$$

$$-2x + 8 + 3y + 6 + z - 2 = 0$$

$$\boxed{-2x + 3y + z + 12 = 0}$$



13. [4+4+4 marks] Given the point $P(2, -6, 8)$ and the lines:

$$l_1: \begin{cases} x = -2 + 3t \\ y = 1 \\ z = 3t \end{cases} \quad \text{and} \quad l_2: \begin{cases} x = s \\ y = -3s \\ z = 3 - 2s \end{cases} \quad s, t \in \mathbb{R}$$

l_1 : thru $R(-2, 1, 0)$
w/ directional vector
 $\vec{v}_1 = (3, 0, 3)$

l_2 : thru $Q(0, 0, 3)$
 $\vec{v}_2 = (1, -3, -2)$

a) compute the shortest distance between them

$$\begin{aligned} 9 - 3t &= -2 \\ -3s &= 1 \\ -2s - 3t &= -3 \end{aligned} \quad \left[\begin{array}{cc|c} 1 & -3 & -2 \\ -3 & 0 & 1 \\ -2 & -3 & -3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & -9 & -5 \\ 0 & -9 & -7 \end{array} \right] \leftarrow \text{no sol'n}$$

$\therefore l_1$ & l_2 do not intersect; $\vec{v}_1 \neq k\vec{v}_2$
 \therefore lines not parallel
 $\Rightarrow l_1$ & l_2 SKEW LINES

$$\vec{RQ} = (2, -1, 3) \quad \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 3 \\ 1 & -3 & -2 \end{vmatrix} = (9, 9, -9) = 9(1, 1, -1)$$

let $\vec{n} = (1, 1, -1)$

$$d = \left\| \text{proj}_{\vec{n}} \vec{RQ} \right\| = \frac{|\vec{n} \cdot \vec{RQ}|}{\|\vec{n}\|} = \frac{|(1, 1, -1) \cdot (2, -1, 3)|}{\sqrt{3}} = \frac{|-2|}{\sqrt{3}} = \frac{2}{\sqrt{3}} \text{ units}$$

b) find the point on l_1 that is closest to the point P

let A be the pt on l_1 closest to P: $A(-2 + 3t, 1, 3t)$
 $\vec{PA} = (-2 + 3t - 2, 1 - (-6), 3t - 8)$
 $= (-4 + 3t, 7, -8 + 3t)$

$$\vec{PA} \cdot \vec{v}_1 = 0: (-4 + 3t, 7, -8 + 3t) \cdot (3, 0, 3) = 0$$

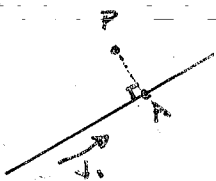
$$= -12 + 9t + 0 - 24 + 9t = 0$$

$$18t = 36$$

$$t = 2$$

$$A(-2 + 3(2), 1, 3(2))$$

$$\boxed{A(4, 1, 6)}$$



c) determine the general equation of the plane containing the line l_1 and the point P

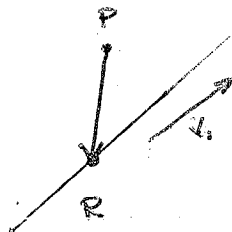
$$\vec{v}_1 = (3, 0, 3) \quad \vec{PR} = (-2 - 2, 1 - (-6), 0 - 8)$$

$$= (-4, 7, -8)$$

$$\vec{v}_1 \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 3 \\ -4 & 7 & -8 \end{vmatrix} = (-21, 12, 21)$$

$$= 3(-7, 4, 7)$$

let $\vec{n} = (-7, 4, 7)$



using P:

$$-7(x - 2) + 4(y + 6) + 7(z - 8) = 0$$

$$-7x + 14 + 4y + 24 + 7z - 56 = 0$$

$$\boxed{-7x + 4y + 7z - 18 = 0}$$

14. [4+4 marks] Let V be the set of 2×2 skew symmetric matrices, i.e. $V = \{A \mid A^T = -A\}$, with the usual operations of addition and scalar multiplication on matrices.

a) Show that V a subspace of M_{22} V not empty $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in V$
 A1: let $B \in V$, $(A+B)^T = A^T + B^T$
 $= -A + (-B)$
 $= -(A+B) \therefore V$ closed under vector addition

S1: (or A6)
 let $K \in \mathbb{R}$, $(KA)^T = KA^T = K(-A) = -K(A) = -(KA)$
 $\therefore V$ closed under scalar multiplication

$\therefore V$ is a subspace of M_{22}

b) Determine a basis and calculate the dimension of the subspace V

$$A \in V \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \Rightarrow \begin{matrix} a = d = 0 \\ b = -c \end{matrix}$$

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

$$\text{basis } \mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\dim(V) = 1$$

15. [4 marks] Let \vec{u} , \vec{v} , and \vec{w} denote vectors in a vector space \mathcal{V} . Show that

$$\text{span}\{\vec{u}, \vec{v}, \vec{w}\} \subseteq \text{span}\{\vec{u} - \vec{v}, \vec{u} + \vec{w}, \vec{w}\}$$

need to show $\exists r, s, t \in \mathbb{F}$

$$i) \vec{u} = r(\vec{u} - \vec{v}) + s(\vec{u} + \vec{w}) + t(\vec{w}) \Rightarrow r = 0, s = 1, t = -1$$

$$ii) \vec{v} = r(\vec{u} - \vec{v}) + s(\vec{u} + \vec{w}) + t(\vec{w}) \Rightarrow r = -1, s = 1, t = -1$$

$$iii) \vec{w} = r(\vec{u} - \vec{v}) + s(\vec{u} + \vec{w}) + t(\vec{w}) \Rightarrow r = 0, s = 0, t = 1$$

$$\therefore \text{span}\{\vec{u}, \vec{v}, \vec{w}\} \subseteq \text{span}\{\vec{u} - \vec{v}, \vec{u} + \vec{w}, \vec{w}\}$$

16. [3+3 marks] Let A be the coefficient matrix of a homogeneous system.

$$A = \begin{bmatrix} 1 & 6 & 4 & 1 & 4 & 3 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

a) Find the solution to the system.

$$\begin{array}{l} r_1: r_1 - 6r_2 \\ \rightarrow \end{array} \begin{bmatrix} 1 & 0 & -2 & -5 & 4 & -15 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1: r_1 - 4r_3} \begin{array}{l} r \\ s \\ t \end{array} \begin{bmatrix} 1 & 0 & -2 & -5 & 0 & -35 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 2r + 5s + 35t$$

$$x_2 = -r - 1s - 3t$$

$$x_3 = r$$

$$x_4 = s$$

$$x_5 = -5t$$

$$x_6 = t$$

Vector eq'n.

$$x = r(2, -1, 1, 0, 0, 0) + s(5, -1, 0, 1, 0, 0) + t(35, -3, 0, 0, -5, 1)$$

b) Find a basis and the dimension for solution space.

$$B = \left\{ (2, -1, 1, 0, 0, 0), (5, -1, 0, 1, 0, 0), (35, -3, 0, 0, -5, 1) \right\}$$

dimension = 3

17. [4marks] For which values of $k \in \mathbb{R}$ is the following linearly independent in \mathbb{P}_2

$$\{1+x, 3x+x^2, 2+x-kx^2\}$$

$$\mathbb{P}_2: a_0 + a_1x + a_2x^2 \quad a_0, a_1, a_2 \in \mathbb{R}$$

$$r(1+x) + s(3x+x^2) + t(2+x-kx^2) = 0 + 0x + 0x^2$$

$$r + 0s + 2t = 0 \quad (\text{coefficients of } x^0)$$

$$r + 3s + t = 0 \quad (\text{coefficients of } x^1)$$

$$0r + s - k = 0 \quad (\text{coefficients of } x^2)$$

trivial sol'n (hence linearly independent in \mathbb{P}_2)
as long as $\det(A) \neq 0$

$$\text{where } A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & -k \end{bmatrix}$$

$$\det(A) = 1(-3k-1) + 2(1)$$

$$= -3k + 1$$

$$\neq 0 \text{ if } \boxed{k \neq \frac{3}{2}, k \in \mathbb{R}}$$