Dawson College Mathematics Department Winter 2010

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Final Examination May 20^{th} 2010
Calculus I (201-103-DW)

STUDENT NAME: ______________________________

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No graphing/programmable calculators allowed.
There are 14 questions in total worth 100 marks.
Show all your work in the space provided and circle your final answers.

[9] **1. Find each of the following limits:**

(a) $\lim_{x\to 5}\frac{x^2+2x-35}{x^2-6x+5}$

$= \lim_{x\to 5} \frac{(x+7)\cancel{(x-5)}}{(x-1)\cancel{(x-5)}}$

$= \frac{12}{4} = 3$

(b) $\lim_{x\to 49}\frac{x-49}{\sqrt{x}-7}$

$= \lim_{x\to 49} \frac{\cancel{(\sqrt{x}-7)}(\sqrt{x}+7)}{\cancel{(\sqrt{x}-7)}}$

$= 14$

(c) $\lim_{x\to\infty}\frac{1+x^2-3x^3}{2x^3+1}$ same degree

$\lim_{x\to\infty} \frac{-3x^3}{2x^3}$

$= -\frac{3}{2}$

[7] **2. Consider the piece-wise defined function:**

$$f(x)=\begin{cases} \dfrac{-x^2}{2}-2x & x\le -1 \\ \dfrac{x}{2}+2 & x>-1 \end{cases}$$

(a) Find $f(-1)$.

$$f(-1) = \frac{-(-1)^2}{2} - 2(-1)$$
$$= \frac{-1}{2} + 2 = \frac{3}{2}$$

(b) Find $\lim_{x\to -1} f(x)$.

$$\lim_{x\to -1^+} = \frac{-1}{2}+2 = \frac{3}{2}$$

$$\lim_{x\to 1^-} = \frac{3}{2}$$

$$\lim_{x\to -1} f(x) = \frac{3}{2}$$

[7] **3. State the limit definition of the derivative (4-step process) and use ONLY the limit definition to evaluate $f'(x)$ for $f(x) = -5x^2 - 4x + 3$.**

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$f'(x) = \lim_{h \to 0} \frac{-5(x+h)^2 - 4(x+h) + 3 - (-5x^2 - 4x + 3)}{h}$$

$$= \lim_{h \to 0} \frac{-5x^2 - 10xh - 5h^2 - 4x - 4h + 3 + 5x^2 + 4x - 3}{h}$$

$$= \lim_{h \to 0} \frac{-10xh - 4h - 5h^2}{h}$$

$$= \lim_{h \to 0} -10x - 4 - 5h \quad (= -10x - 4)$$

[4] **4. Consider the function $f(x) = \frac{-5}{3}x^2 + 2x + 2$. Find the equation of the tangent line to the graph of this function at $x = -1$.**

$$f'(x) = -\frac{10}{3}x + 2$$

$$m = f'(-1) = \frac{-10}{3}(-1) + 2$$

$$= \frac{10}{3} + \frac{6}{3}$$

$$= \frac{16}{3}$$

$$y = f(-1) = \frac{-5}{3}(-1)^2 + 2(-1) + 2$$

$$y = mx + b$$

$$-\frac{5}{3} = \frac{16}{3}(-1) + b$$

$$\frac{-5}{3} + \frac{16}{3} = b$$

$$\frac{11}{3} = b$$

$$\boxed{y = \frac{16}{3}x + \frac{11}{3}}$$

[16] **5. Find the derivative for each function. (You do not have to simplify.)**

(a) $f(x)=\left(\dfrac{2x-1}{x^2+x}\right)^4$

$$f'(x) = 4\left(\frac{2x-1}{x^2+x}\right)^3 \cdot \left[\frac{(x^2+x)(2) - (2x-1)(2x+1)}{(x^2+x)^2}\right]$$

(b) $f(x)=\sin(x^3)-\sqrt{\cos(2x)}+e^{\tan x}$

$f'(x) = \cos(x^3)\cdot 3x^2$ ... $e^{\tan x}$

$2\sqrt{\cos(2x)}$

$$= 3x^2\cos(x^3) + \frac{\sin(2x)}{\sqrt{\cos(2x)}} + e^{\tan x}\sec^2 x$$

(x^2+3)

$- \ln(x^2+2) - 1 \ln(1-x).$

$$f'(x) = \frac{2x}{x^2+3} + \frac{1}{|2(1-x)|}$$

[5] 6. Find $\frac{dy}{dx}$ given $y = x^{\sqrt{x}}$.

$$\ln y = \sqrt{x} \ln x$$

$$\frac{1}{y} y' = \sqrt{x} \frac{1}{x} + \ln x \cdot \frac{1}{2} x^{-1/2}$$

$$\frac{1}{y} y' = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$$

$$y' = x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right)$$

[6] 7. Find $f''(2)$ if $f(x) = \sqrt{x^3 - 4}$. (Find the value of the second derivative when $x = 2$.)

$$f'(x) = \frac{1}{2}(x^3-4)^{-1/2} \cdot 3x^2$$

$$= \frac{3}{2}x^2 (x^3-4)^{-1/2}$$

(a) Find the marginal cost function and use it to estimate the cost of producing the 11-th unit.

$$C'(x) = 3x^2 - 18x + 33$$

$$C'(10) = 3(100) - 18(10) + 33$$

$$= 153 \$$$

[6] 9. **Consider the relation:** $x^3 + y^3 = 6xy$.

(a) Find the derivative $\frac{dy}{dx}$.

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

$$3x^2 - 6y = \frac{dy}{dx}(6x - 3y^2)$$

$$\frac{3x^2 - 6y}{6x - 3y^2} = \frac{dy}{dx}$$

$$\hookrightarrow \boxed{\frac{dy}{dx} = \frac{x^2 - 2y}{2x - y^2}}$$

(b) Find the equation of the tangent line to the graph of this relation at the point $(3,3)$.

$\underline{y = mx + b}$

$$m = \frac{(3)^2 - 2(3)}{2(3) - (3)^2}$$

$$= \frac{3}{-3} = -1$$

$\underline{y = -x + b}$

$$3 = -3 + b$$

$$6 = b$$

$$\boxed{y = -x + 6}$$

[5] 10. **Consider the function $f(x) = (x^2 - 1)^3$. Find the absolute maximum and minimum values for $f(x)$ on the closed interval $[-1, 2]$.**

$$f'(x) = 3(x^2-1)^2 \cdot 2x$$

$$0 = 6x(x^2-1)^2$$

$$x = 0 \qquad x = \pm 1$$

$$f(-1) = ((-1)^2-1)^3 = 0$$

$$\boxed{f(0) = (-1)^3 = -1}$$ absolute min

$$f(1) = 0$$

$$\boxed{f(2) = 27}$$ absolute max.

[4] 11. **Find the x-values where the graph of the function $f(x) = \dfrac{(x-1)^2}{(2x+1)^3}$ has a horizontal tangent line.**

$$f'(x) = \frac{(2x+1)^3 2(x-1) - (x-1)^2 3(2x+1)^2 \cdot 2}{}$$

$$= \frac{2(x-1)(2x+1)^2\left[(2x+1) - 3(x-1)\right]}{(2x+1)^6}$$

$$0 = \frac{-2(x-1)(x-4)}{(2x+1)^4}$$

[12] 12. Consider the function $f(x) = x^4 - 4x^3$.

(a) Find the x and y intercepts.

$(0,0)$

$0 = x^3(x-4)$

$(0,0)$, $(4,0)$

(b) Find the intervals of increase/decrease and the relative extrema.

$f'(x) = 4x^3 - 12x^2$

$0 = 4x^2(x-3)$

$x=0$ $\quad$ $x=3$

dec $\quad$ dec $\quad$ inc

f' $\quad$ $-$ $\quad$ $-$ $\quad$ $+$

$-\infty$ $\quad$ 0 $\quad$ 3 $\quad$ ∞

local min

$(3,-27)$

(c) Find the intervals of concavity and the points of inflection.

$f''(x) = 12x^2 - 24x$

UP $\quad$ DOWN $\quad$ UP

[7] 13. **A manufacturer estimates that x units of its product can be produced at a total cost of $C(x)=x^3+100x+45000$ dollars. If the total revenue from the sale of x units is $R(x)=4600x$ dollars, determine the level of production**

that will maximize the profit. Also, calculate the maximum profit.

$P(x) = R(x) - C(x)$

$= 4600x - x^3 - 100x - 45'000$

$= -x^3 + 4500x - 45\,000$

$P'(x) = -3x^2 + 4500$

$0 = -x^2 + 1500$

$x^2 = 1500$

$x = +38.73 \simeq 39$ units.

$\left.\begin{array}{l} P'(2) = + \\ P'(40) = - \end{array}\right\}$ $\boxed{\text{max at } x = 39}$

$P(39) = -59319 + 175'500 - 45'000$

[6] **14. Find the antiderivatives:**

(a) $\int \frac{(x^4 + 3\sqrt{x})}{x^2}\,dx$

$$= \int \frac{x^4}{x^2}dx + 3\int \frac{x^{1/2}}{x^2}\,dx$$

$$= \int x^2 dx + 3\int x^{-3/2}\,dx$$

$$= \frac{x^3}{3} + 3\,x^{-1/2}(-2) + C$$

$$= \frac{x^3}{3} - \frac{6}{\sqrt{x}} + C$$

(b) $\int x^2\sqrt{x^3+1}\,dx$

$$\left.\begin{aligned} u &= x^3+1 \\ du &= 3x^2\,dx \end{aligned}\right\}$$

$$= \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\,u^{3/2}\cdot\frac{2}{3} + C$$

$$= \frac{2}{9}(x^3+1)^{3/2} + C$$