

DAWSON COLLEGE
Mathematics Department
Final Examination
Calculus II
Commerce
201-NYB-05

Thursday, May 19, 2011

Student Name _____

Student I.D. # _____

Instructors: M. Ishii, A. Jimenez, M. Marchant, I. Rajput
Sections: 13, 14, 15, 16

TIME: 9:30 am – 12:30 pm.

Question #	Marks
1/25	

Instructions:

1.) (25 Marks)

Calculate the following integrals

a) $\int \cos(\sqrt{x}) dx = \int \cos(u) 2\sqrt{x} du$ (4 Marks)

$$dx = 2\sqrt{x} dy$$

b) $\int \sec^2(3x) \tan^3(3x) dx = \int \sec^2(u) u^3 du$ (5 Marks)

c) $\int \frac{2x-3}{\sqrt{x-2}} dx = \int \frac{2u+1}{\sqrt{u}} du = \int 2\sqrt{u} + \frac{1}{\sqrt{u}} du$ (6 Marks)

$$= \frac{4}{3} \sqrt{x-2}^3 + 2\sqrt{x-2} + C$$

e) $\int \frac{x-1}{x^2(x-3)} dx$

(5 Marks)

$$\frac{A}{x} + \frac{D}{x^2} + \frac{C}{x-3} = \frac{Ax(x-3) + D(x-3) + Cx^2}{x^2(x-3)}$$

$$x-1 = Ax(x-3) + D(x-3) + Cx^2$$

$$x=0$$

$$-1 = D(-3) \quad B = \frac{1}{3}$$

$$x=3$$

$$2 = C3^2 \Rightarrow C = \frac{2}{9}$$

$$x=1$$

$$0 = A(-2) + \frac{1}{3}(-2) + \frac{2}{9} \Rightarrow A(-2) = \frac{4}{9} \Rightarrow A = -\frac{2}{9}$$

$$-\frac{2}{9} \ln|x| - \frac{1}{3} \frac{1}{x} + \frac{2}{9} \ln|x-3| + K$$

2.)

(5 Marks)

Monthly records show that the rate of change for the cost of a product is

$$C'(x) = 3\sqrt{2x+25}$$

and the fixed cost for a month is \$11 125. What is the total cost of producing 300 items in a month?

$$C(x) = \int C'(x) dx = \int 3\sqrt{2x+25} dx = \frac{3}{2} \int \sqrt{u} du$$

$$u = 2x + 25$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{1}{2} du$$

$$= \frac{3}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \sqrt{u}^3$$

$$= \sqrt{2x+25}^3 + C$$

When $C(0) = 11\ 125$

$$\sqrt{2 \cdot 0 + 25}^3 + C = 5 + C = 11\ 125$$

Ans

$$\begin{array}{r} \$26\ 745 \\ \hline 15625 + 11120 \end{array}$$

$$C(300) = \sqrt{2 \cdot 300 + 25}^3 + 11\ 120 = \sqrt{625}^3 + 11\ 120$$

Find the average value of the function $f(x) = xe^{-x^2}$ over the interval $[-1, 1]$

$$F(x) = \int xe^{-x^2} dx = \int x e^{2u} \frac{1}{-2x} dx = -\frac{1}{2} e^{-x^2}$$

$$u = -x^2$$

$$\frac{du}{dx} = -2x$$

$$dx = \frac{1}{-2x} du$$

$$\bar{f} = \frac{F(1) - F(-1)}{1 - (-1)} = \frac{-\frac{1}{2}e^{-1} + \frac{1}{2}e^{-1}}{2} = \frac{0}{2} = 0$$

4.) (4 Marks)

Calculate the following integral using only the (Riemman-sums) definition

$$\int_0^1 (2x - x^2) dx$$

Ans: $\frac{2x^2}{2} - \frac{x^3}{3} \Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$ Ans

$$S_n = \sum_{k=0}^n \left(\frac{2k}{n} - \frac{k^2}{n^2} \right) \frac{1}{n} = \sum_{k=0}^n \frac{2k}{n^2} - \sum_{k=0}^n \frac{k^2}{n^3}$$

$$= \frac{2}{n^2} \frac{n(n+1)}{2} - \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$S_n = \frac{n+1}{n} - \frac{1}{6} \frac{n+1}{n} \frac{2n+1}{n}$$

$$\lim_{n \rightarrow \infty} S_n = 1 - \frac{1}{6} \cdot 1 \cdot 2 = 1 - \frac{1}{3} = \frac{2}{3}$$

5.) (5 Marks)

Find the total area of the region enclosed by the functions $f(x) = \sqrt{7x+15}$ and $g(x) = x+3$

$$f(x) = g(x)$$

$$7x+15 = x^2 + 6x + 9$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3 \quad x = -2$$

$$\left(\begin{array}{l} f(0) = \sqrt{15} \\ g(0) = 3 \\ f > g \\ \text{on } [-2, 3] \end{array} \right.$$

$$F(x) = \int \sqrt{7x+15} - x - 3 \, dx = \frac{2}{21} \sqrt{7x+15}^3 - \frac{x^2}{2} - 3x$$

$$F(3) - F(-2) = \frac{2}{21} \sqrt{36}^3 - \frac{3^2}{2} - 3 \cdot 3 =$$

$$- \frac{2}{21} \sqrt{7(-2)+15}^3 + \frac{(-2)^2}{2} + 3(-2)$$

6.) (4 Marks)

Use Simpson's rule to approximate $\int_0^2 \sqrt{1+x^2} dx$ with $n=4$. The calculations should be done with four decimals precision.

$$x_0 = 0 \quad x_1 = \frac{1}{2} \quad x_2 = 1 \quad x_3 = \frac{3}{2} \quad x_4 = 2$$

$$\Delta x = \frac{2}{4} = \frac{1}{2}$$

$$\int_0^2 \sqrt{1+x^2} dx \approx \frac{\Delta x}{3} \left(1 + 4\sqrt{1+\frac{1}{4}} + 2\sqrt{1+1} + 4\sqrt{1+\frac{9}{4}} + \sqrt{1+5} \right)$$

$$= \frac{2}{3} \left(1 + 2\sqrt{5} + 2\sqrt{2} + 2\sqrt{13} + \sqrt{6} \right)$$

$$\approx \frac{2}{3} \left(1 + 4.47213 + 2.82842 + 7.21111 + 2.44948 \right)$$

$$= \frac{2}{3} \cdot 17.96114 = 11.97409$$

7.)

(8 Marks)

Calculate the following limits

$$a) \quad \lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{x^2} = \frac{9(\cos(0))}{2} = \frac{9}{2} \quad (4 \text{ Marks})$$

$$\frac{[1 - \cos(3x)]'}{[x^2]'} = \frac{3\sin(3x)}{2x} =$$

$$\frac{[3\sin(3x)]'}{[2x]'} = \frac{9\cos(3x)}{2}$$

b) $\lim_{x \rightarrow \infty} \frac{e^{4x} + x}{1 + e^{5x}} = \lim_{x \rightarrow \infty} \left(\frac{4}{5} \frac{1}{e^x} + \frac{1}{5e^{5x}} \right) = 0$ (4 Marks)

9.)

(4 Marks)

An insurance company is receiving monthly payments of 42 000 dollars. Find the future value of

that company 10 years from now with an annual interest rate of 8% compounded continuously.

$$FV = e^{0.08 \cdot 10} \int_0^{10} 42000 \cdot 12 e^{-0.08t} dt$$

Or annuity formula

$$FV = \frac{12 \cdot 42000}{0.08} (e^{0.08 \cdot 10} - 1)$$

$$= 150 \cdot 42000 (e^{0.8} - 1)$$

$$\approx 6300000 (2.2255 - 1)$$

10.)
(5 Marks)

Find the particular solution of the differential equation of $y' = \frac{\ln x}{y}$ subject to the condition that $y(e) = 2$

$$\int y \, dy = \int \ln x \, dx \quad \text{by parts}$$

$$\frac{y^2}{2} = x \ln x - x + C$$

$$\frac{e^2}{2} = e \ln e - e + C$$

$$\frac{e^2}{2} = C$$

or

$$y^2 = 2x \ln x - 2x + C$$

$$C = e^2$$

etc

$$C = C$$

11.)

(4 Marks)

Find the fourth degree Taylor polynomial of the function

$$f(x) = \ln(1-x) \quad \text{for values of } x \text{ close to } 0$$

$$f(x) = \ln(1-x) \rightarrow f(0) = \ln(1) = 0$$

$$f'(x) = -\frac{1}{1-x} \rightarrow f'(0) = -1$$

$$f''(x) = \frac{1}{(1-x)^2} \rightarrow f''(0) = 1$$

$$f'''(x) = -\frac{2}{(1-x)^3} \rightarrow f'''(0) = -2$$

$$f^{(4)}(x) = \frac{6}{(1-x)^4} \rightarrow f^{(4)}(0) = 6$$

$$P_4(x) = -x + \frac{x^2}{2} - \frac{1}{3}x^3 + \frac{1}{4}x^4$$

12.)

(4 Marks)

Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{2+3^{n+1}}{5^n} = 10$$

$$\begin{aligned} 2 \sum \left(\frac{1}{5}\right)^n + 3 \sum \left(\frac{3}{5}\right)^n &= \frac{2}{1-\frac{1}{5}} + \frac{3}{1-\frac{3}{5}} \\ &= \frac{2 \cdot 5}{4} + \frac{3 \cdot 5}{2} = \frac{5}{2} + \frac{3 \cdot 5}{2} = \frac{4 \cdot 5}{2} = 10 \end{aligned}$$

13.)

(12 Marks)

Show whether the following series converges or diverges

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+n}$

(4 Marks)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+n} < \sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2} = \sum_{n=1}^{\infty} \frac{1}{4n^{3/2}} \quad p = \frac{3}{2} > 1$$

Converges Converges

b) $\sum_{n=1}^{\infty} \frac{5n^2-3n}{7-4n^2}$

Diverges

(4 Marks)

$$\lim_{n \rightarrow \infty} \frac{5n^2-3n}{7-4n^2} = -\frac{5}{4} \neq 0$$

c) $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$

Diverges

(4 Marks)

$$u = 1 + \ln n$$

$$\frac{du}{dn} = \frac{1}{n}$$

$$dn = n du$$

$$f(n) = \int \frac{1}{n} \frac{1}{u} n du = \ln|u| = \ln|1 + \ln n|$$

$$\lim_{n \rightarrow \infty} \ln|1 + \ln n| = \infty$$

$$f(1) = \ln(1+0) = 0$$

14.)

(5 Marks)

The demand function for a certain commodity is $p = D(x) = -0.6x^2 + 160$ and the supply function is $p = S(x) = 0.4x^2 + 50$

15.)

(4 Marks)

An investment is expected to generate income at the rate of $R(t) = 1000$ dollars per year for the next 20 years. Find the present and future value of this investment if the interest is at 5% per year compounded continuously.

$$FV = e^{0.05 \cdot 20} \int_0^{20} 1000 e^{-0.05 \cdot t} dt =$$

= or annuity function

$$FV = \frac{mP}{r} (e^{rT} - 1) = \frac{20 \cdot 1000}{0.05} (e^{0.05 \cdot 20} - 1)$$

$$= 400000 (e - 1) = 687312.73$$

INFORMATION SHEET

$$\sum_{k=1}^n 1 = n$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)] \quad \checkmark$$

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n)] \quad \checkmark$$

$$A = FV = e^{rT} \int^T R(t) e^{-rt} dt$$

$$PV = \int^T R(t) e^{-rt} dt$$

$$CS = \int_0^{\bar{x}} D(x) dx - p\bar{x}$$

$$PS = p\bar{x} - \int_0^{\bar{x}} S(x) dx$$

$$E(L_n) = \sum^n \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$
